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Complete set of infinitesimal transformations about n-soliton solutions of the KdV equation

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Abstract. Using the Backlund transformations, we have obtained infinitesimal transformations about an n-soliton solution of the KdV equation whch form a complete set. They are shown to be the squares of eigenfunctions of the Schrödinger equation with the n-soliton as the potential. It is also shown that the conserved densities for an n-soliton are n in number.

1. Introduction

The infinitesimal transformation (IT) about any solution $U(x, t)$ of the equation

$$U_t = K(U) \tag{1.1}$$

is defined as a function $Y(x, t)$ such that $U(x, t)+\varepsilon Y(x, t)$, $\varepsilon \ll 1$, is also a solution of (1.1). K is, in general, a nonlinear operator. Wadati (1978) has obtained a countable infinity of IT about the solution of the KdV equation and established a one-to-one correspondence between the IT and the infinite number of conservation laws. Rubinstein (1970) has obtained IT about a one-soliton solution of the sine–Gordon (SG) equation which form a complete set and studied the linear stability of a soliton. Fogel *et al* (1977) have used this complete set of IT to study various physical phenomena giving rise to perturbed SG equations.

Case (1978) has shown that one IT can be obtained about a solution of a general nonlinear equation from each conserved density. The countable infinity of conserved densities for a KdV equation gives a countable infinity of IT about any solution. These IT, however, would not form a complete set. This will be seen for the n-soliton solution from the results of this paper. In this case this countable infinity of IT consists of linear combinations of n IT and these do not form a complete set. As will be shown, a continuum of IT about an n-soliton together with the n IT mentioned above form a complete set. The complete set of IT about a one-soliton solution of the SG equation (Rubinstein 1970) also consists of one discrete bounded IT and a continuum of IT. The countable infinity of IT about any solution of the SG equation degenerates to this one discrete IT for a one-soliton solution.

In this paper, using the Backlund transformation (BT) of the KdV equation (Wahlquist and Estabrook 1973), we construct IT about $W_{1,n}(x, t)$ (where the n-soliton solution of the KdV equation $U_{1,n}(x, t) = -[W_{1,n}(x, t)]_x$, the subscript x denoting a partial derivative) which form a complete set. The IT are shown to be the squares of eigenfunctions of the Schrödinger equation with the n-soliton as the potential. Further, the IT are eigenfunctions of an operator $T(U)$. Generally $T(U)$ acting on an IT about $W(x, t)$ ($U(x, t) = -W_x(x, t)$ is a solution of the KdV equation) gives another IT about $W(x, t)$.

We show that, about an n-soliton, the countable infinity of IT obtained by Wadati (1978) consists of linear combinations of n IT about the n-soliton. It then follows that there are only n conserved quantities for an n-soliton.

Since the expressions for the IT are obtained recursively in this paper, we see how the eigenfunctions of the Schrödinger equation change when the potential changes from an $(n-1)$-soliton to an n-soliton.

In § 2, we derive the differential equation for the IT about a 'new' solution $W'(x, t)$ of (2.2) in terms of IT about an 'old' solution $W(x, t)$. In § 3, after fixing the notations, IT about $W_{1,n}(x, t)$ are derived. It is shown that IT can be obtained from the solution of a second-order differential equation. In § 4, the IT are shown to be equal to the squares of the eigenfunctions of the Schrödinger equation. The completeness of the IT is established. In § 5 we show that there are only n conserved quantities for an n-soliton.

2. Differential equation for the IT

Following Wahlquist and Estabrook (1973) we consider the KdV equation

$$U_t + 12UU_x + U_{xxx} = 0. \tag{2.1}$$

With $U(x, t) = -W_x(x, t)$, the equation for $W(x, t)$ is

$$W_t - 6(W_x)^2 + W_{xxx} = 0. \tag{2.2}$$

The IT $Y(x, t)$ about $W(x, t)$ satisfies the equation

$$Y_t + 12UY_x + Y_{xxx} = 0. \tag{2.3}$$

The first half of the BT is (Wahlquist and Estabrook 1973)

$$W'_x + W_x = (W' - W)^2 - k^2. \tag{2.4}$$

The differential equation for $Z(x, t)$, an IT about $W'(x, t)$, in terms of $Y(x, t)$, an IT about $W(x, t)$, is, from (2.4),

$$Z_x + Y_x = 2(W' - W)(Z - Y). \tag{2.5}$$

The second half of the BT and the corresponding equations for $Z(x, t)$ and $Y(x, t)$ are needed only to show that $Z(x, t)$ obtained from (2.5) is an IT about $W'(x, t)$ and are not given here. Equation (2.5) is the basic equation to derive the IT.

For $W(x, t) = 0$, a solution of (2.2), (2.3) reduces to

$$Y_t + Y_{xxx} = 0. \tag{2.6}$$

Its solutions are

$$Y = \exp[i(\kappa x + \omega t)] \tag{2.7}$$

with the dispersion relation

$$\omega = \kappa^3.$$

We use (2.5) repeatedly, starting with $W(x, t) = 0$ and $Y(x, t)$ given by (2.7), to obtain the IT about $W_{1,n}(x, t)$.

3. IT about $W_{1,n}(x, t)$

3.1. Notation

We represent an n-soliton with parameters $k_1, k_2, \ldots, k_n$ by $U_{1,n}$. $U_{1,n}(r)$ will represent an $(n-1)$-soliton with parameters k_p, $1 \leq p \leq n$, $p \neq r$. $U_{1,n}(r, s)$ will similarly represent an $(n-2)$-soliton. A one-soliton with parameter k_p will be represented, for convenience, by U_p instead of $U_{1,n}(1, 2, \ldots, p-1, p+1, \ldots, n)$. $W_{1,n}$, $W_{1,n}(r)$, $W_{1,n}(r, s)$ and W_p are given by $U_{1,n} = -(W_{1,n})_x$, etc.

3.2. Theory

Equation (2.5) is written in a form suitable for soliton solutions:

$$Z_x + Y_x = 2(W_{1,n} - W_{1,n}(r))(Z - Y). \tag{3.1}$$

Here $Y(x, t)$ is an IT about $W_{1,n}(r)$ and $Z(x, t)$ an IT about $W_{1,n}$.

Define a function $\alpha_{1,n}(\kappa)$ of (x, t) recursively by

$$\alpha_{1,n}(\kappa) = [2(W_{1,n} - W_{1,n}(r_1)) + \mathrm{i}\kappa]\alpha_{1,n}(\kappa; r_1) + 2(\alpha_{1,n}(\kappa; r_1))_x \tag{3.2a}$$

where

$$\alpha_{1,n}(\kappa; r_1) = [2(W_{1,n}(r_1) - W_{1,n}(r_1, r_2)) + \mathrm{i}\kappa]\alpha_{1,n}(\kappa; r_1, r_2) + 2(\alpha_{1,n}(\kappa; r_1, r_2))_x \tag{3.2b}$$

and

$$\alpha_p(\kappa) = 2W_p + \mathrm{i}\kappa \tag{3.3}$$

where for convenience we write $\alpha_{1,n}(\kappa; 1, 2, \ldots, p-1, p+1, \ldots, n)$ as $\alpha_p(\kappa)$. κ takes all real values. We shall show that the IT about $W_{1,n}$, $W_{1,n}(r)$, etc, are related to $\alpha_{1,n}(\kappa)$, $\alpha_{1,n}(\kappa; r)$, etc.

It is easy to verify that $\alpha_p(\kappa)$ satisfies the differential equation

$$\alpha_{xx} + \mathrm{i}\kappa\alpha_x + 2U_p\alpha = 0. \tag{3.4}$$

Assuming $\alpha_{1,n}(\kappa; r)$ satisfies

$$\alpha_{xx} + \mathrm{i}\kappa\alpha_x + 2U_{1,n}(r)\alpha = 0 \tag{3.5}$$

and using

$$U_{1,n} + U_{1,n}(r) = k_r^2 - (W_{1,n} - W_{1,n}(r))^2, \tag{3.6}$$

the result obtained by differentiating it and (3.2a) we can show that $\alpha_{1,n}(\kappa)$ satisfies the equation

$$\alpha_{xx} + \mathrm{i}\kappa\alpha_x + 2U_{1,n}\alpha = 0. \tag{3.7}$$

One of its solutions is obtained from (3.2) and (3.3). The other solution will be discussed in the sequel.

We shall now show that

$$Y_{1,n}(\kappa) = (\exp[\mathrm{i}(\kappa x + \omega t)])(\alpha_{1,n}(\kappa))^2 \tag{3.8}$$

is an IT about $W_{1,n}$ obtained by successive applications of (3.1), starting from the IT $Y = \exp[\mathrm{i}(\kappa x + \omega t)]$ about the zero solution of (2.2).

With

$$Y_r(\kappa) = (\exp[\mathrm{i}(\kappa x + \omega t)])(\alpha_r(\kappa))^2 \tag{3.9}$$

$Y_r(\kappa)/(\kappa^2 + 4k_r^2)$ is a solution of (3.1) with $W_{1,n} = W_r$ (so that $W_{1,n}(r) = 0$), $Y = \exp[\mathrm{i}(\kappa x + \omega t)]$ (an IT about the zero solution $W_{1,n}(r) = 0$) and $k = k_r$.

Thus $Y_r(\kappa)$ is an IT about $W_r(x, t)$. Using (3.5) and (3.6) it can be verified that $Y_{1,n}(\kappa)/(\kappa^2 + 4k_r^2)$ is a solution of (3.1) with

$$Y = Y_{1,n}(\kappa; r) = (\exp[\mathrm{i}(\kappa x + \omega t)])(\alpha_{1,n}(\kappa; r))^2. \tag{3.10}$$

Therefore our contention is proved. Here $Y_{1,n}(\kappa; r)$ is an IT about $W_{1,n}(r)$. We wish to remark that $\alpha_{1,n}(\kappa)$ and hence $Y_{1,n}(\kappa)$ is independent of the value of r, $1 \leqslant r \leqslant n$, in (3.2). The proof is inductive. It is straightforward but lengthy and is not reproduced here.

Henceforward we omit the term $\exp(\mathrm{i}\omega t)$ with the understanding that it is always present with $\exp(\mathrm{i}\kappa x)$.

The second linearly independent solution of (3.7), $\bar{\alpha}_{1,n}(\kappa)$, is given by

$$\bar{\alpha}_{1,n}(\kappa) = \exp(-\mathrm{i}\kappa x)\alpha_{1,n}(-\kappa). \tag{3.11}$$

This can be proved by transforming the dependent variable in (3.7),

$$\alpha = \exp(-\mathrm{i}\kappa x)\beta. \tag{3.12}$$

The IT obtained by using (3.11) in (3.8) is

$$\begin{aligned} \bar{Y}_{1,n}(\kappa) &= (\exp(\mathrm{i}\kappa x))(\bar{\alpha}_{1,n}(\kappa))^2 \\ &= (\exp(-\mathrm{i}\kappa x))(\alpha_{1,n}(-\kappa))^2 = Y_{1,n}(-\kappa). \end{aligned} \tag{3.13}$$

Since κ takes all real values we do not get any new IT from the linearly independent solution $\bar{\alpha}_{1,n}(\kappa)$.

3.3. *IT for complex values of κ—the discrete case*

In the last subsection we considered κ real. However (3.8) with $\kappa = 2\mathrm{i}k_r$, $1 \leqslant r \leqslant n$, are also IT about $W_{1,n}$.

To see this, we find using (3.2), (3.5) and (3.6) that with $\kappa = 2\mathrm{i}k_r$ $\alpha_{1,n}(2\mathrm{i}k_r)$ satisfies

$$\alpha_x/\alpha = W_{1,n} - W_{1,n}(r) + k_r. \tag{3.14}$$

Therefore from (3.8) and (3.14)

$$\begin{aligned} Y_{1,n}(2\mathrm{i}k_r) &= (\exp(-2k_r x))(\alpha_{1,n}(2\mathrm{i}k_r))^2 \\ &= \exp\left(2\int (W_{1,n} - W_{1,n}(r))\,\mathrm{d}x\right). \end{aligned} \tag{3.15}$$

On the other hand, if we put $Y = 0$ in (3.1), we obtain

$$Z = \exp\left(2\int (W_{1,n} - W_{1,n}(r))\,\mathrm{d}x\right). \tag{3.16}$$

Comparing (3.15) and (3.16) we see that $Y_{1,n}(2\mathrm{i}k_r)$ are the IT about $W_{1,n}$ obtained by starting from the identity IT about $W_{1,n}(r)$. There are n IT corresponding to k_r, $1 \leqslant r \leqslant n$.

Using (3.6) and the following result due to Wahlquist and Estabrook (1973),

$$W_{1,n} = W_{1,n}(r, s) - \frac{k_s^2 - k_r^2}{W_{1,n}(r) - W_{1,n}(s)}, \tag{3.17}$$

we can evaluate the integral in (3.15). Thus an explicit expression can be obtained for $Y_{1,n}(2ik_r)$:

$$k_r^2 Y_{1,n}(2ik_r) = U_r/[(W_{1,n}(r) - W_{1,n}(r_1))^2(W_{1,n}(r, r_1) - W_{1,n}(r_1, r_2))^2 \dots \times (W_{1,n}(r, r_1, \dots, r_{n-2}) - W_{1,n}(r_1, r_2, \dots, r_{n-1}))^2] \tag{3.18}$$

where $(r_1, r_2, \dots, r_{n-1})$ is some permutation of $(1, 2, \dots, r-1, r+1, \dots, n)$. We thus get $(n-1)!$ different forms for $Y_{1,n}(2ik_r)$.

The other linearly independent solution of (3.7) for $\kappa = 2ik_r$ diverges and will not give an IT.

3.4. The operator $T(U)$ and its eigenfunctions

We shall show that $Y_{1,n}(\kappa)$ are eigenfunctions of the operator

$$T(U) = -\left(\frac{1}{4}\frac{\partial^2}{\partial x^2} + 2U - \int^x dz\, U_z\right) \tag{3.19}$$

with eigenvalues $\frac{1}{4}\kappa^2$, where κ is real or $\kappa = 2ik_r$, $1 \leqslant r \leqslant n$.

The result is obviously true for the IT $\exp(i\kappa x)$ about the zero solution. With $U = 0$ in (3.19),

$$T(0)(\exp(i\kappa x)) = \tfrac{1}{4}\kappa^2 \exp(i\kappa x). \tag{3.20}$$

It is tedious but direct to show that if $Z(x, t)$ is a solution of (2.5) for a $Y(x, t)$ then $T(U')[Z]$ is a solution of (2.5) with $Y(x, t)$ replaced by $T(U)[Y]$. Assume now that

$$T(U_{1,n}(r))(Y_{1,n}(r, \kappa)) = \tfrac{1}{4}\kappa^2 Y_{1,n}(r, \kappa). \tag{3.21}$$

Substituting for $Y(x, t)$ in (3.1), the RHS and the LHS successively, and using the result stated above equation (3.21) we obtain

$$T(U_{1,n})(Y_{1,n}(\kappa)) = \tfrac{1}{4}\kappa^2 Y_{1,n}(\kappa). \tag{3.22}$$

For $\kappa = 2ik_r$ we have

$$T(U_{1,n})(Y_{1,n}(2ik_r)) = -k_r Y_{1,n}(2ik_r). \tag{3.23}$$

The result is proved.

The adjoint $T^+(U)$ of $T(U)$ is

$$T^+(U) = -\left(\frac{1}{4}\frac{\partial^2}{\partial x^2} + 2U + U_x \int^x dz\right) \tag{3.24}$$

and for any $f(x)$

$$T^+(U)(f_x(x)) = -(\tfrac{1}{4}f_{xxx} + 2Uf_x + U_x f) \tag{3.25}$$

which is the recursion relation obtained by Wadati (1978) for IT about any solution $U(x, t)$ of the KdV equation. Any differences in the coefficients in (3.25) and in the work of Wadati (1978) arise because we have a factor 12 in the KdV equation (2.1) instead of

6. We have done this so as to use the results of Wahlquist and Estabrook (1973) without any change.

From the fact that $T^{+}(U)$ generates IT about $U(x, t)$ and the relation

$$\partial[T(U)(f(x))]/\partial x = T^{+}(U)(f_x(x)) \tag{3.26}$$

it follows that $T(U)$ acting on an IT about $W(x, t)$ generates another IT about $W(x, t)$.

4. Relation of (3.7) to the Schrödinger equation and completeness of IT

Using (3.7) it is seen that

$$\psi = (\exp[\tfrac{1}{2}\mathrm{i}(\kappa x)])\alpha_{1,n}(\kappa) \tag{4.1}$$

satisfies the Schrödinger equation

$$\psi_{xx} + (2U_{1,n} + \tfrac{1}{4}\kappa^2)\psi = 0. \tag{4.2}$$

The squares of the two independent solutions ψ and $\bar{\psi}$, for a given κ real, are, using (3.8) and (4.1),

$$\psi^2 = Y_{1,n}(\kappa) \tag{4.3a}$$

$$\bar{\psi}^2 = Y_{1,n}(-\kappa). \tag{4.3b}$$

Since $W_{1,n}$ is invariant to change of sign of k_r we see from (3.15) that

$$Y_{1,n}(2\mathrm{i}k_r) = Y_{1,n}(-2\mathrm{i}k_r). \tag{4.4}$$

Thus for the discrete eigenvalues $\kappa = 2\mathrm{i}k_r$, from (4.3) and (4.4),

$$\bar{\psi}^2|_{\kappa=-2\mathrm{i}k_r} = \psi^2|_{\kappa=2\mathrm{i}k_r} = \psi\bar{\psi}|_{\kappa=\pm 2\mathrm{i}k_r} = Y_{1,n}(2\mathrm{i}k_r). \tag{4.5}$$

It has been shown by Kaup (1976) that $\psi^2(\kappa, x)$, $\bar{\psi}^2(\kappa, x)$, $\psi^2|_{\kappa=2\mathrm{i}k_r}$, $\bar{\psi}^2|_{\kappa=-2\mathrm{i}k_r}$, $\psi\bar{\psi}|_{\kappa=\pm 2\mathrm{i}k_r}$ form a complete set. Combining this result with (4.3) and (4.5), we have $Y_{1,n}(\kappa)$ for all real κ and $\kappa = 2\mathrm{i}k_r$, $1 \leqslant r \leqslant n$, form a complete set.

Thus one could obtain a complete set of IT about $W_{1,n}$ by solving (4.2). $U_{1,n}$ is a reflectionless potential and (4.2) has been solved by Kay and Moses (1956). On the other hand, the method described here using the BT is an alternative way of obtaining the eigenfunctions of (4.2). It has the advantage of being a recursive definition and we see how the eigenfunctions of (4.2) change when the potential changes from an $(n-1)$-soliton to an n-soliton.

Combining the results of this section and § 3.4 we see that the squares of the eigenfunctions of the Schrödinger equation, with $U_{1,n}$ as potential, are eigenfunctions of $T(U_{1,n})$. That this result is true for any solution of $U(x, t)$ has been shown by Ablowitz *et al* (1974) and Newell and Flaschka (1975). In these cases, however, explicit expressions for the eigenfunctions cannot be obtained.

5. Finiteness of conserved densities for n-soliton

Gardner *et al* (1974) have shown that

$$U_{1,n} = -4 \sum_{m=1}^{n} k_m \psi_m^2 \tag{5.1}$$

where ψ_m is an eigenfunction of the Schrödinger equation with $U_{1,n}$ as the potential and $(-k_m^2)$ as the eigenvalue. A generalisation of (5.1) to other nonlinear equations and to non-soliton solutions has been given by Newell (1978). From (4.5) and (5.1) it follows that $U_{1,n}$ is a linear combination of the n IT corresponding to the n discrete eigenvalues of (4.2).

From (3.23), (3.26) and (4.5) it follows that $(\psi_m^2)_x$, $m = 1, \ldots, n$, are eigenfunctions of $T^+(U_{1,n})$. Now $(T^+(U_{1,n}))^p$, $p = 0, 1, \ldots, \infty$, acting on $(U_{1,n})x$ generate the countable infinity of IT about $U_{1,n}$ obtained by Wadati (1978). Differentiating (5.1) with respect to x and applying $(T^+(U_{1,n}))^p$ on both sides we see that the countable infinity of IT about $U_{1,n}$ consists of all linear combinations of the n functions $(\psi_m^2)_x$, $m = 1, \ldots, n$. Since there is a conservation law associated with $(T^+(U_{1,n}))^p(U_{1,n})_x$, for each p, it follows that there are only n conserved densities for the n-soliton.

A special case of this result has been proved by Gardner *et al* (1974). They have shown that, for the one-soliton, all conserved densities are proportional to the one-soliton.

6. Conclusions and comments

Using the BT we have obtained the IT about an n-soliton of the KdV equation. These IT are shown to be squares of eigenfunctions of the Schrödinger equation with potential $U_{1,n}$. The IT are shown to form a complete set. Since the expression for the IT is recursive, we know how the eigenfunctions change as the potential in the Schrödinger equation changes from an $(n-1)$-soliton to an n-soliton. Further it has been shown that the number of conserved quantities for an n-soliton are n in number.

It is possible to extend the results of this paper to other nonlinear equations. We have explicitly obtained (Aiyer 1981) the IT about the one-soliton solution of the SG equation in terms of the squares of the eigenfunctions of the corresponding Zakharov–Shabat equations. These results agree with those obtained by Rubinstein (1970). This also shows that the basis functions chosen for the solution of the perturbed SG equation by Fogel *et al* (1977) on the one hand and by Kaup and Newell (1978) on the other are related.

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